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ECE 310 §E

Midterm Note Sheet
ECE 310
Fall 2017

BIBO Stability

- Absolute Summability - pz plot - $|x[n]| < \alpha \Rightarrow |y[n]| < \beta$

$\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]h[n-m] = \sum_{m=-\infty}^{+\infty} h[m]x[n-m] = x[n] * h[n]$$

for LSI systems

z - Transform

$$X(z) = \sum_{n=0}^{+\infty} x[n]z^{-n}$$

Delay Property	$nx[n] \longleftrightarrow -z\left(\frac{dX(z)}{dz}\right)$
$y[n-k]u[n-k]$	Multiplication by n
$\Leftrightarrow z^{-k}Y(z)$	

Include ROCs of transforms

DTFT only defined when ROC includes unit circle

DTFT

$$X_d(\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} \qquad x(t) = \int_{-\infty}^{\infty} X_d(\omega)e^{j\omega t}d\omega$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega n}d\omega$$

$$H_d(\omega) = H(z)\big|_{z=e^{j\omega}}$$

Real systems

Magnitude - even symmetry
Phase - odd symmetry

$X_d(\omega) = X_d^*(-\omega)$

$$x[n] = \cos(\omega_0 n + \theta) \longrightarrow y[n] = |H_d(\omega_0)|\cos(\omega_0 n + \theta + \angle H_d(\omega_0))$$

Eigensequence Property

$$e^{j\omega_0 n} \rightarrow \boxed{H(\omega)} \rightarrow H(\omega_0)e^{j\omega n}$$

$$\cos(\omega_0 n) \rightarrow \boxed{H(\omega)} \rightarrow |H(\omega_0)|\cos(\omega_0 n + \angle H(\omega_0))$$

$$1 + e^{j\omega_0 n} = e^{-j\frac{\omega_0}{2}n}\left(e^{j\frac{\omega_0}{2}n} + e^{-j\frac{\omega_0}{2}n}\right) = 2\cos\left(\frac{\omega_0}{2}n\right)e^{-j\frac{\omega_0}{2}n}$$

Geometric Sums

$$\sum_{n=0}^{N} r^n = \frac{1-r^{n+1}}{1-r}$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad (|r|<1)$$

Parseval's relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X_d(\omega)|^2 d\omega$$ for aperiodic signals w/ finite energy

Linearity

$x_1[n] \rightarrow y_1[n]$

$x_2[n] \rightarrow y_2[n]$

$\Rightarrow ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$

Shift Invariance

$x[n] \longrightarrow y[n]$

$\Rightarrow x[n-n_0] \longrightarrow y[n-n_0]$

Causality

Output cannot depend on future input values
outside circle in z-transform

		outside	on UC	inside
$\{p_\ell^{(H)}\}$	single			✓
	repeated			✓
$\{p_\ell^{X}\}$	single		✓	✓
	repeated			✓
$\{p_\ell^{Y}\}$	single		✓	✓
	repeated			✓

Delta function

Kronecker

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

Dirac

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{unbounded} & t = 0 \end{cases}$$

s.t. $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

$\delta'(-t) = -\delta'(t)$
derivative odd

$$\mathrm{sinc}(x) = \frac{\sin x}{x}$$

Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\omega = \frac{e^{j\omega} + e^{-j\omega}}{2}$$

$$\sin\omega = \frac{e^{j\omega} - e^{-j\omega}}{2j}$$

Special case	
- LSI	h : L
- stable	x : M
- real	y : $L+M-1$

Z-Transform Pairs

$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$(\cos\omega_0 n)u[n]$	$\frac{1-(\cos\omega_0 n)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$(\sin\omega_0 n)u[n]$	$\frac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$(r^n\cos\omega_0 n)u[n]$	$\frac{1-(r\cos\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > r$
$(r^n\sin\omega_0 n)u[n]$	$\frac{(\sin\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > r$

$\sin(a+b) = \sin a\cos b + \cos a\sin b$

$\sin(a-b) = \sin a\cos b - \cos a\sin b$

$\cos(a+b) = \cos a\cos b - \sin a\sin b$

$\cos(a-b) = \cos a\cos b + \sin a\sin b$

DTFT Pairs

$1 \longleftrightarrow 2\pi\delta(\omega),\ |\omega| \le \pi$ or $2\pi\sum_{\ell=-\infty}^{+\infty}\delta(\omega+2\ell\pi)$

$e^{j\omega_0 n} \longleftrightarrow 2\pi\delta(\omega-\omega_0)$

$\cos\omega_0 n \longleftrightarrow \pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$

$\sin\omega_0 n \longleftrightarrow \frac{\pi}{j}(\delta(\omega-\omega_0)-\delta(\omega+\omega_0))$

$\delta[n] \longleftrightarrow 1$

$u[n] \longleftrightarrow \frac{1}{1-e^{-j\omega}} + \pi\delta(\omega)$

$a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}},\ |a|<1$

Realness

$h[n]$

LCCDE: $y[n] + \sum_{\ell=1}^{N} a_\ell\, y[n-\ell] = \sum_{\ell=0}^{N} b_\ell\, x[n-\ell]$

every value real
a, b real

$$H(z) = \frac{\sum_{\ell=0}^{N} b_\ell z^{-\ell}}{1+\sum_{\ell=0}^{N} a_\ell z^{-\ell}}$$

poles: real or
conjugate pairs

$H_d(\omega)$

mag even
phase odd

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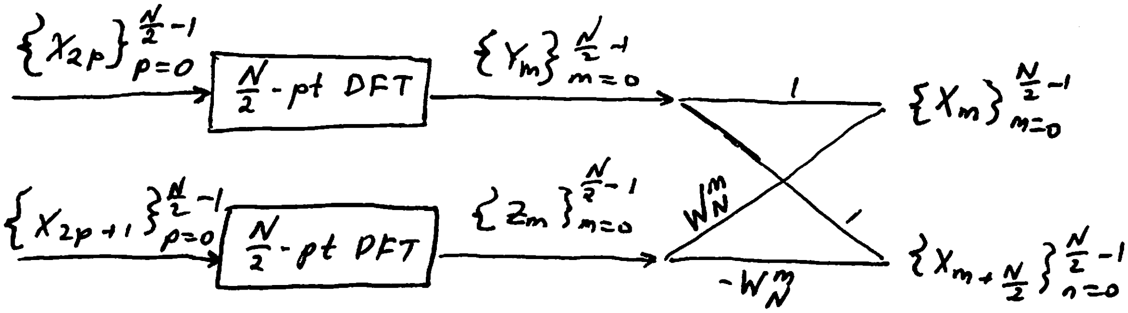
FFT Butterfly



$$\begin{cases} X_m = Y_m + W_N^m Z_m \\ X_{m+\frac{N}{2}} = Y_m - W_N^m Z_m \end{cases}$$

$$W_N^m = \left(e^{\frac{-j2\pi}{N}}\right)^m$$
Twitch factor

Y_m : $\frac{N}{2}$ point DFT
$Y[\langle m+\frac{N}{2}\rangle_{N/2}] = Y[m]$
$X[\langle m+N\rangle_N] = X[\langle m\rangle]$

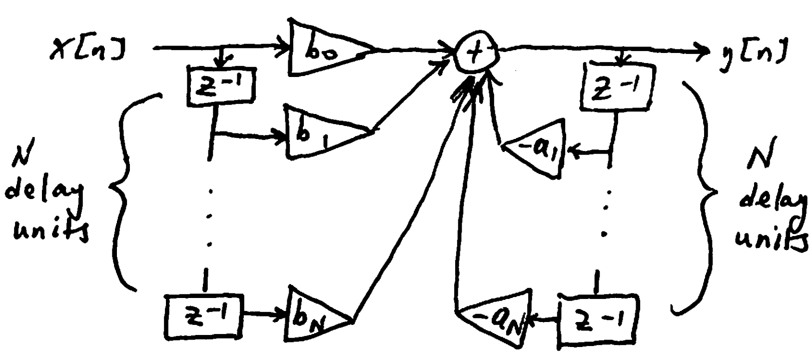
Fast Convolution

$$\hat{y}[n] = \text{IFFT}\{\text{FFT}\{\hat{x}[n]\}\cdot \text{FFT}\{\hat{h}[n]\}\}$$

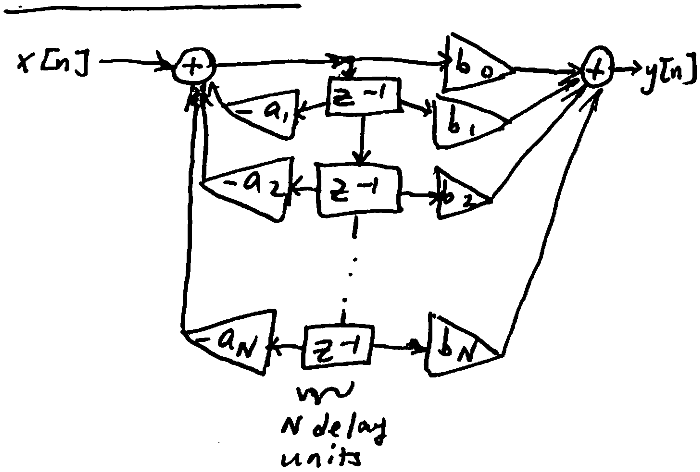
$$L = M+N-1$$

$$\{y[n]\}_{n=0}^{L-1} = \{x[n]\}_{n=0}^{L-1} \circledast \{h[n]\}_{n=0}^{L-1} = \sum_{\ell=0}^{L-1} x[\ell]\, h[\langle n-\ell\rangle_L]$$

Direct Form I



Direct Form II



Generalized Linear Phase (GLP)

Type			Symmetry	Length	LPF	HPF	BPF	BSF
1	I	$\alpha = 0, \pm\pi$	Even	Odd	Y	Y	Y	Y
	II		Even	Even	Y	X	Y	X
2	III	$\alpha = \pm\frac{\pi}{2}$	odd	odd	X	X	Y	X
	IV		odd	Even	X	Y	Y	X

$h[n] = \pm h[N-1-n]$

LCCDE:

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{N} b_k x[n-k]$$

Transfer function:

$$H(z) = \frac{\sum_{k=0}^{N} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

Filter Design by Windowing

$$d_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} D(\omega)\, e^{j\omega n}\, d\omega$$

1. Find $D_d(\omega)$ – desired response
2. Determine symmetry, length, window type
3. Create $G_d(\omega) = D_d(\omega)\, e^{j(\alpha - M\omega)}$, $M = \frac{N-1}{2}$ $g[n] = DTFT^{-1}\{G_d(\omega)\}$
4. Find $g[n] = DTFT^{-1}\{G_d(\omega)\}$
5. Apply window $w[n]$

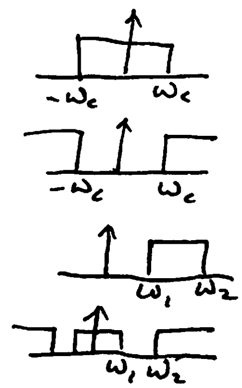
Before adding window

LPF $\quad g[n] = \frac{\omega_c}{\pi}\,\text{sinc}\left(\omega_c\left(n - \frac{N-1}{2}\right)\right)$

HPF $\quad g[n] = (-1)^n\, \frac{\pi - \omega_c}{\pi}\,\text{sinc}\left((\pi - \omega_c)\left(n - \frac{N-1}{2}\right)\right)$

BPF $\quad g[n] = \frac{\omega_2}{\pi}\,\text{sinc}\left(\omega_2\left(n - \frac{N-1}{2}\right)\right) - \frac{\omega_1}{\pi}\,\text{sinc}\left(\omega_1\left(n - \frac{N-1}{2}\right)\right)$

BSF $\quad g[n] = \frac{\omega_2}{\pi}\,\text{sinc}\left(\omega_2\left(n - \frac{N-1}{2}\right)\right) - \frac{\omega_1}{\pi}\,\text{sinc}\left(\omega_1\left(n - \frac{N-1}{2}\right)\right)$



Windows

Window name	Side lobe level (dB)	Approx $\Delta\omega$	Exact $\Delta\omega$	$\delta_p \approx \delta_s$	A_p (dB)	A_s (dB)
Rectangular	−13	$4\pi/L$	$1.8\pi/L$	0.09	0.75	21
Bartlett	−25	$8\pi/L$	$6.1\pi/L$	0.05	0.45	26
Hann	−31	$8\pi/L$	$6.2\pi/L$	0.0063	0.055	44
Hamming	−41	$8\pi/L$	$6.6\pi/L$	0.0022	0.019	53
Blackman	−57	$12\pi/L$	$11\pi/L$	0.0002	0.0017	74

Symmetry

$$w[n] = \begin{cases} w[M-n] & 0 \le n \le M \\ 0 & \text{othewise} \end{cases} \quad M = L-1$$

Rectangular/Boxcar/Truncation

$$w[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Hamming

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Bartlett (triangular)

$$w[n] = \begin{cases} 2n/(N-1), & 0 \le n \le \frac{N-1}{2}, \quad N-1 \text{ even} \\ 2 - 2n/(N-1), & \frac{N-1}{2} \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Blackman

$$w[n] = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Hann

$$w[n] = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Kaiser window – I_0

– adjustible to achieve specification depending on tradeoff

– works best for amount of energy in side lobes

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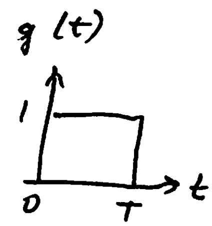
$$\omega = \Omega T$$

Zero Order Hold (ZOH)

Approximation to the ideal D/A

$x[n] \rightarrow \boxed{D/A} \rightarrow x(t)$
ideal

$x[n] \rightarrow \boxed{ZOH} \rightarrow x'(t) \neq x(t)$

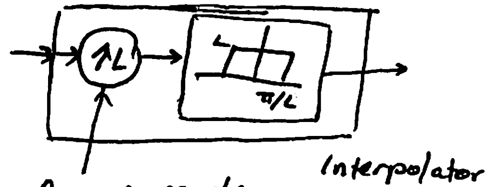


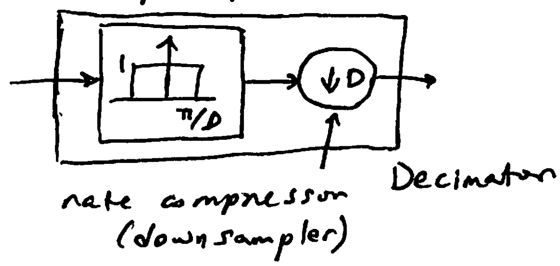
$$x[n] \rightarrow \boxed{ZOH} \rightarrow y(t) = \sum_{n=-\infty}^{\infty} x[n]\, g(t - nT)$$

$$G_a(\Omega) = T e^{-j\frac{\Omega T}{2}} \operatorname{sinc}\left(\frac{\Omega T}{2}\right)$$
CTFT of analog ZOH

$$Y_a(\Omega) = G_a(\Omega)\, Y_d(\Omega T)$$

Multirate DSP



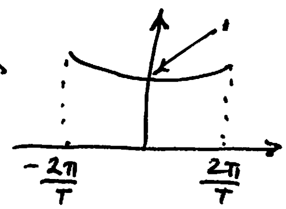


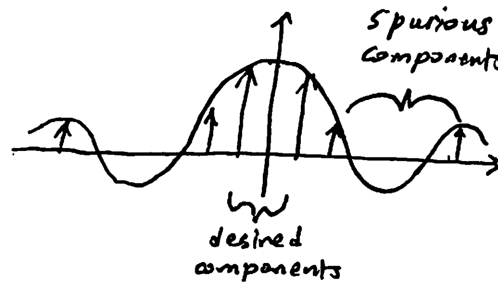
Compensation Filter

ZOH is not ideal

$y[n] \rightarrow \underbrace{\boxed{ZOH} \rightarrow \boxed{F_a(\Omega)}}_{\text{Ideal D/A}} \rightarrow y(t)$

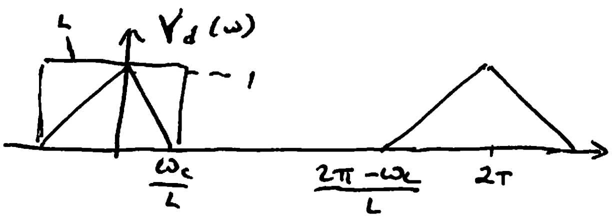
Anti-Aliasing Filter



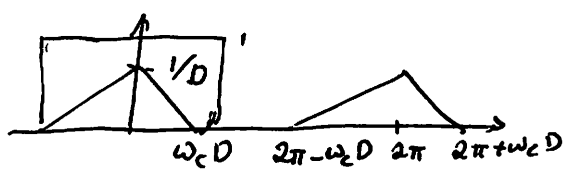


$X_a(t) \xrightarrow[\text{A/D (ideal)}]{T} \underbrace{\circledast \uparrow L \rightarrow \boxed{H(\omega)\ \pi/L}}_{\text{Interpolator}} \rightarrow \underbrace{\boxed{H_d(\omega)}}_{\text{DSP}} \rightarrow \underbrace{\boxed{H(\omega)\ \pi/D} \rightarrow \downarrow D}_{\text{Decimator}} \rightarrow \underbrace{\boxed{ZOH} \rightarrow \boxed{F_a(\Omega)}}_{\text{Ideal D/A}} \rightarrow y_a(t)$

↑L

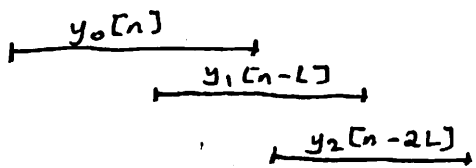


↓D



Overlap & Add

$$\{x[n]\}_{n=0}^{\infty}, \quad \{h[n]\}_{n=0}^{M-1}$$



$\{x_0(\ell)\}_{\ell=0}^{L-1}$ $\{x_1(\ell)\}_{\ell=0}^{L-1}$ . . . $\{x_k(\ell)\}_{\ell=0}^{L-1}$ . . .

$$x[n] = \sum_{k=0}^{\infty} x_k(n - kL)$$

$$y[n] = x[n] * h[n]$$

$$y[n] = h[n] * \sum_{k=0} x_k(n - kL) = \sum_{k=0} h[n] * x_k(n - kL)$$

Eigen functions (of LTI systems)

LTI system $H_d(\omega)$

$$e^{j\omega_0 n} \rightarrow \boxed{H_d(\omega)} \rightarrow e^{j\omega_0 n} H_d(\omega_0)$$

for $H_d(\omega)$ real $\longleftrightarrow$ $H_d(\omega) = H_d^*(-\omega)$

$$\cos(\omega_0 n + \phi) \rightarrow \boxed{H_d(\omega)} \rightarrow |H_d(\omega_0)| \cos(\omega_0 n + \phi + \measuredangle H_d(\omega_0))$$

For non-real systems, break up cosine:

$$\cos(\omega_0 n + \phi) \Rightarrow \frac{e^{j\omega_0 n} e^{j\phi} + e^{-j\omega_0 n} e^{-j\phi}}{2}$$

IIR Filter Design

dB: $20\log|H_d(\omega)|$

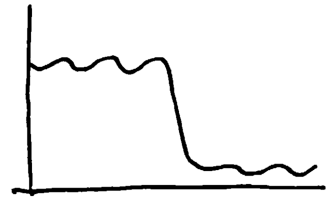
Butterworth: maximally flat response in ~~pass~~ band and stopband

Chebyshev I: optimum in the minimax sense. over passband

Equiripple passband and monotone decreasing stopband

Chebyshev II: optimum in the minimax sense. over stopband

Equiripple stopband and monotone decreasing passband

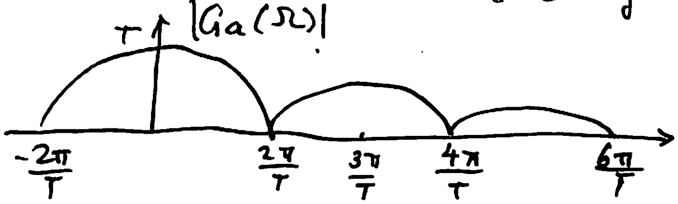


Elliptical ("Cauer"): equiripple passband and stopband

Bessel filter – best phase response

optimal – smallest order

sinc



Discrete Fourier Transform (DFT)

DFT: $X[m] \triangleq \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}mn}$ $m = 0 \ldots N-1$, $n = 0 \ldots N-1$

DFT^{-1}: $x[n] = \frac{1}{N} \sum_{m=0}^{N-1} X[m]\, e^{+j\frac{2\pi}{N}mn}$

$$X[m] \triangleq X_d(\omega)\Big|_{\omega = \frac{2\pi}{N}m} = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}mn}$$

$$\{x_n\}_{n=0}^{N-1} \rightarrow \boxed{\text{DFT}} \rightarrow \{X[m]\}_{n=0}^{N-1}$$

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Real Multiplications and Additions

Linear convolution

M multiplies } per input sample
M-1 additions }

"Fast convolution"

w/ reduced butterflies

ea. FFT and IFFT takes $\frac{K}{2}\log_2 K$ multiplies, $K\log_2 K$ additions

$\frac{3K}{2}\log_2 K + K = K(1+\frac{3}{2}\log_2 K)$ multiplies } per input sample
$3K\log_2 K$ additions }

Overlap & Add

length L frames

$K = L+M-1 \nearrow 2^?$

one length-K FFT, one length-K complex multiplications, one length K inverse FFT

$2\frac{K}{2}\log_2 K + K = K(1+\log_2 K)$ ~~per~~ complex multiplies per frame

$\left\lceil \frac{N}{K-M+1} \right\rceil K(1+\log_2 K) + \frac{K}{2}\log_2 K$ ~~pe~~ complex multiplies per input sample

N - length of $x[n]$ - large!

multiply by 4 to get real multiplies

FIR vs IIR

- If GLP is desired, an FIR is the only option
- An IIR will require fewer multiplier, adder, and delays to implement a filter achieving given magnitude freq. response specs, than an FIR